DEGENERATION OF A-INFINITY MODULES

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ABSTRACT. In this paper we use A_∞-modules to study the derived category of a finite dimensional algebra over an algebraically closed field. We study varieties parameterising A_∞-modules. These varieties carry an action of an algebraic group such that orbits correspond to quasi-isomorphism classes of complexes in the derived category. We describe orbit closures in these varieties, generalising a result of Zwara and Riedtmann for modules.

INTRODUCTION

In this paper we study parameter spaces of objects in the derived category of a finite dimensional algebra Λ over an algebraically closed field k.

The derived category of Λ can be defined by formally inverting all quasi-isomorphisms in the category of complexes of Λ-modules. Recall that a quasi-isomorphism is a chain map which induces isomorphism in homology. So two isomorphic objects in the derived category have isomorphic homologies, but the converse is in general not true. A natural point of view is therefore that quasi-isomorphism classes are determined by the isomorphism type of their homology and some extra information which explains how the homology is put together to form a complex. In this paper we will take this point of view.

We study families of objects in the derived category with fixed dimensions or even fixed isomorphism type in homology. We do so by replacing the ordinary derived category with the equivalent derived category of A_∞-modules [K]. The equivalence functor preserves homology, so dimensions and isomorphism types are not affected. The A_∞-modules are called polydules in [L-H], and we use both names in this paper.

Keller has observed [K] that by fixing dimensions in homology $\underline{d} = (d_a, \ldots, d_b)$ we obtain a variety, here denoted by $\mathcal{M} = \mathcal{M}_{\Lambda,\underline{d}}$, which parameterises polydules with homology dimensions $\underline{d}$. There is an algebraic group $\mathcal{G} = \mathcal{G}_{\underline{d}}$ acting on $\mathcal{M}_{\Lambda,\underline{d}}$ such that orbits correspond to quasi-isomorphism classes of polydules. It is this variety and group action we examine in the present paper.

The main result is a generalisation of a result of Zwara [Z] and Riedtmann [R] from module varieties to these varieties of polydules.

Theorem. *Let $M, N \in \mathcal{M}_{\Lambda,\underline{d}}$. Then $N \in \overline{\mathcal{G} \cdot M}$ if and only if there is a finite dimensional Λ-polydule Z and an exact triangle*

$$Z \to M \oplus Z \to N \to Z[1].$$

The paper is organised as follows. In section 1 we recall the definition of A_∞-modules and their morphisms. In section 2 we give a summary of the most important known results concerning A_∞-modules and their derived category. In section 3 we describe in detail the variety $\mathcal{M}$ and the group $\mathcal{G}$ with its action on $\mathcal{M}$. We consider the special cases where the homology is given as representations of a quiver with relations or where the homology module structure is fixed in section 4. In section 5 we prove our main result, an algebraic characterisation of degeneration. We give several examples in section 6.

1. A-infinity modules

In this section we give the basic definitions of A_∞-modules and their morphisms. We stress that our definitions are adapted to the case we are interested in, namely A_∞-modules over associative algebras, and are only valid for this case. For the definitions in the most general case (A_∞-modules over A_∞-algebras), we refer to [K] and [L-H].

For our purposes we find it most practical to use a left module notation. A (left) A_∞-module over an algebra Λ in our sense is the same as a (right) A_∞-module over Λ^{op} in the sense of [K]. Since all our algebra elements are in degree 0 we avoid certain sign issues.

1.1. **Modules.** Let k be a commutative ring and Λ an associative k-algebra. An A_∞-*module* over Λ (or a Λ-*polydule* [L-H]) is a graded k-module

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

with k-linear graded maps

$$m_n \colon \Lambda^{(\otimes_k)n-1} \otimes_k M \to M, \ n \geq 1$$

of degree $n - 2$ satisfying the rules

$$m_1 m_1 = 0,$$
$$m_1 m_2 = m_2(\mathbf{1} \otimes m_1),$$

and for $n \geq 3$

$$\sum_{i=1}^{n}(-1)^{i(n-1)}m_{n-i+1}(\mathbf{1}^{\otimes n-i} \otimes m_i) = \sum_{j=1}^{n-2}(-1)^{j-1}m_{n-1}(\mathbf{1}^{\otimes n-j-2} \otimes \mu \otimes \mathbf{1}^{\otimes j}),$$

where $\mathbf{1}$ is the identity map and μ is the multiplication in Λ.

In particular (M, m_1) is a complex of k-modules. Note that if M is bounded, then $m_n = 0$ for $n \gg 0$.

1.2. **Change of base rings.** Suppose $M = \bigoplus_{i \in \mathbb{Z}} M^i$ a Λ-polydule with maps m_n^M, $n \geq 1$. If V is an (ungraded) k-module, then the graded k-module

$$M \otimes_k V = \bigoplus_{i \in \mathbb{Z}}(M^i \otimes_k V)$$

is a Λ-polydule with maps

$$m_n^{M \otimes V} = m_n^M \otimes \mathbf{1}_V \colon \Lambda^{(\otimes_k)n-1} \otimes_k M \otimes_k V \to M \otimes_k V, \ n \geq 1.$$

Suppose now that $\theta\colon k \to k'$ is a morphism of commutative rings. Then $\Lambda \otimes_k k'$ is a k'-algebra, and a $\Lambda \otimes_k k'$-polydule is a graded k'-module $N = \bigoplus_{i \in \mathbb{Z}} N^i$ with k'-linear maps

$$m_n^N \colon (\Lambda \otimes_k k')^{(\otimes_{k'})n-1} \otimes_{k'} N \to N, \ n \geq 1$$

satisfying the equations. Since

$$(\Lambda \otimes_k k')^{(\otimes_{k'})n-1} \otimes_{k'} N \simeq \Lambda^{(\otimes_k)n-1} \otimes_k N,$$

this structure is determined by k'-linear maps

$$m_n^N \colon \Lambda^{(\otimes_k)n-1} \otimes_k N \to N, \ n \geq 1.$$

The map $\theta\colon k \to k'$ induces a Λ-polydule structure on N.

If $M = \bigoplus_{i \in \mathbb{Z}} M^i$ a Λ-polydule, there is a $\Lambda \otimes_k k'$-polydule structure on the graded k'-module

$$M \otimes_k k' = \bigoplus_{i \in \mathbb{Z}} (M^i \otimes_k k')$$

given by the k'-linear maps

$$m_n^{M \otimes k'} = m_n^M \otimes \mathbf{1}\colon \Lambda^{(\otimes_k)n-1} \otimes_k M \otimes_k k' \to M \otimes_k k', \ n \geq 1.$$

1.3. Morphisms and compositions.
A *morphism* $f\colon L \to N$ between two Λ-polydules L and N is given by a family of maps

$$f_n \colon \Lambda^{(\otimes_k)n-1} \otimes_k L \to N, \ n \geq 1$$

of degree $n-1$ satisfying the rules

$$f_1 m_1 = m_1 f_1,$$
$$f_1 m_2 - f_2(\mathbf{1} \otimes m_1) = m_2(\mathbf{1} \otimes f_1) + m_1 f_2,$$

and for $n \geq 3$

$$\sum_{i=1}^n (-1)^{i(n-1)} f_{n-i+1}(\mathbf{1}^{\otimes n-i} \otimes m_i) + \sum_{j=1}^{n-2} (-1)^j f_{n-1}(\mathbf{1}^{\otimes n-j-2} \otimes \mu \otimes \mathbf{1}^{\otimes j})$$
$$= \sum_{r=1}^n (-1)^{(r+1)n} m_{n-r+1}(\mathbf{1}^{\otimes n-r} \otimes f_r).$$

Note in particular that f_1 is a chain map $f_1\colon (L, m_1) \to (N, m_1)$ between complexes of k-modules. A morphism is called a *quasi-isomorphism* if $f_1\colon (L, m_1) \to (N, m_1)$ is a quasi-isomorphism of complexes.

A morphism f is called *strict* if $f_i = 0$ for $i > 1$. Furthermore f is called a *strict monomorphism* if f is strict and f_1 is a monomorphism. Similarly f is called a *strict epimorphism* if f is strict and f_1 is an epimorphism.

The *identity morphism* $\mathbf{1}_N\colon N \to N$ is given by $f_1 = \mathbf{1}$ and $f_i = 0$ for all $i > 0$. The *composition* $fg\colon N \to M$ of two morphisms $f\colon L \to M$ and $g\colon N \to L$ is given by the rule

$$(fg)_n = \sum_{i=1}^n (-1)^{(i-1)n} f_{n-i+1}(\mathbf{1}^{\otimes n-i} \otimes g_i).$$

A morphism $f\colon L \to N$ is called *null-homotopic* if there is a family of maps

$$h_n\colon \Lambda^{(\otimes_k)n-1} \otimes_k L \to N, \ n \geq 1$$

of degree $-n$ such that

$$f_1 = m_1 h_1 + h_1 m_1,$$
$$f_2 = m_1 h_2 - m_2(1 \otimes h_1) - h_2(1 \otimes m_1) + h_1 m_2$$

and for $n \geq 3$

$$f_n = \sum_{r=1}^{n} (-1)^{r(n-1)} m_{n-r+1}(1^{\otimes n-r} \otimes h_r)$$
$$+ \sum_{i=1}^{n} (-1)^{i(n-1)} h_{n-i+1}(1^{\otimes n-i} \otimes m_i)$$
$$+ \sum_{j=1}^{n-2} (-1)^j h_{n-1}(1^{\otimes n-j-2} \otimes \mu \otimes 1^{\otimes j}).$$

To morphisms f and g from L to N are called *homotopic* if $f - g$ is null-homotopic. In this case we say that h is an *homotopy* between f and g. A morphism $f\colon L \to N$ is called a *homotopy equivalence* if there exists a morphism $g\colon N \to L$ such that gf is homotopic to 1_L and fg is homotopic to 1_N.

1.4. **Unitality.** A Λ-polydule M is called *strictly unital* if for all $a \in M$ $m_2(1, a) = a$ and $m_n(\lambda_1, \ldots, \lambda_{n-1}, a) = 0$ if $n \geq 3$ and $1 \in \{\lambda_1, \ldots, \lambda_{n-1}\}$.

A morphism $f\colon L \to N$ is called *strictly unital* if $f_n(\lambda_1, \ldots, \lambda_{n-1}, a) = 0$ whenever $n \geq 2$ and $1 \in \{\lambda_1, \ldots, \lambda_{n-1}\}$. Similarly an homotopy h between two morphisms f and g is called *strictly unital* if $h_n(\lambda_1, \ldots, \lambda_{n-1}, a) = 0$ whenever $n \geq 2$ and $1 \in \{\lambda_1, \ldots, \lambda_{n-1}\}$. If there is such a homotopy between f and g, we denote this fact by $f \sim_h g$.

2. The derived category

Suppose now that k is a field and Λ a k-algebra. In this section we give a summary of the, for our purposes, most important results from [K] and [L-H] concerning A_∞-modules and their derived category.

Let $\mathrm{Mod}_\infty \Lambda$ denote the category of strictly unital Λ-polydules and strictly unital morphisms. The *homotopy category* $\mathcal{H}_\infty(\Lambda)$ is defined to be the quotient category $\mathcal{H}_\infty(\Lambda) = \mathrm{Mod}_\infty \Lambda / \sim_h$, so morphisms in $\mathcal{H}_\infty(\Lambda)$ are homotopy classes of morphisms in $\mathrm{Mod}_\infty \Lambda$. The following theorem is of central importance.

Theorem 2.1. [K, 4.2] [L-H, 2.4.1.1] *A morphism $f\colon L \to N$ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

This gives us one out of several essentially equivalent ways of defining the *derived category* $\mathcal{D}_\infty(\Lambda)$ (see section 4.1.3 in [L-H]). We simply define it to be the same as the homotopy category, so $\mathcal{D}_\infty(\Lambda) = \mathcal{H}_\infty(\Lambda)$.

The category $\mathcal{D}_\infty(\Lambda)$ is a triangulated category. The *shift* of a Λ-polydule M, denoted $M[1]$, is the Λ-polydule defined by $(M[1])^i = M^{i+1}$ and $m_n^{M[1]} =$

$(-1)^n m_n^M$. If $f\colon L \to N$ is a morphism in $\mathcal{D}_\infty(\Lambda)$, then there exists an object cone f, called the *cone* of f, and an exact triangle

$$L \xrightarrow{f} N \xrightarrow{j} \text{cone } f \xrightarrow{p} L[1]$$

in $\mathcal{D}_\infty(\Lambda)$. The cone has graded parts

$$(\text{cone } f)^i = N^i \oplus (L[1])^i,$$

and the Λ-polydule structure is defined by

$$m_n^{\text{cone } f} = \begin{pmatrix} m_n^N & (-1)^n f_n \\ 0 & m_n^{M[1]} \end{pmatrix}.$$

Every exact triangle in $\mathcal{D}_\infty(\Lambda)$ is isomorphic to one of the form above. The morphism j is a strict monomorphism, the morphism p is a strict epimorphism and the sequence of graded k-vector spaces $0 \to N \xrightarrow{j_1} \text{cone } f \xrightarrow{p_1} L[1] \to 0$ is exact.

Using the rotation axiom, we get the same description of exact triangles as in [K, 5.2]. Let $L \xrightarrow{j} M \xrightarrow{p} N$ be a sequence with j a strict monomorphism, with p a strict epimorphism and where the sequence of graded k-vector spaces $0 \to L \xrightarrow{j_1} M \xrightarrow{p_1} N \to 0$ is exact. We call such a sequence an *exact sequence of Λ-polydules*. In this case there is an exact triangle $L \xrightarrow{j} M \xrightarrow{p} N \to L[1]$ in $\mathcal{D}_\infty(\Lambda)$, so cone $j \simeq N$ in $\mathcal{D}_\infty(\Lambda)$. All exact triangles are obtained in this way.

If (C, δ) is a complex of Λ-modules, then C can be viewed as a Λ-polydule where $m_1 = \delta$, the map $m_2\colon \Lambda \otimes C \to C$ is given by the Λ-module structure and $m_n = 0$ for $n \geq 3$. Identifying complexes with polydules in this way induces a triangle functor $\mathcal{D}(\text{Mod } \Lambda) \to \mathcal{D}_\infty(\Lambda)$, where $\mathcal{D}(\text{Mod } \Lambda)$ is the usual (unbounded) derived category of Λ. It turns out that this functor is an equivalence.

Theorem 2.2. [K, 4.3] [L-H, 2.4.2.3] *The canonical functor $\mathcal{D}(\text{Mod } \Lambda) \to \mathcal{D}_\infty(\Lambda)$ is an equivalence of triangulated categories.*

In other words each quasi-isomorphism class of Λ-polydules corresponds to (has as a subclass) exactly one quasi-isomorphism class of complexes of Λ-modules. So every polydule is quasi-isomorphic to a complex in this sense.

In this paper however, we will focus on another type of polydules, the ones with $m_1 = 0$. A good thing about such objects is that quasi-isomorphisms between them are invertible already in $\text{Mod}_\infty \Lambda$.

Theorem 2.3. [K, 4.3] *Suppose M and N are two Λ-polydules with $m_1^M = 0 = m_1^N$. Then M and N are quasi-isomorphic if and only they are isomorphic.*

Each quasi-isomorphism class of polydules has an object with $m_1 = 0$, so polydules of this type can be chosen as representatives of isomorphism classes in the derived category $\mathcal{D}_\infty(\Lambda)$.

Theorem 2.4. [L-H, 3.3.1.7] *Suppose M is a strictly unital Λ-polydule. Then there is a strictly unital A_∞-module M' with $m_1^{M'} = 0$ and a strictly unital quasi-isomorphism $M' \to M$.*

If M is a complex, meaning $m_n^M = 0$ for $n \geq 3$, then the graded parts of M' are the homology modules of (M, m_1^M). In general if N is a Λ-polydule with $m_1^N = 0$, then each N^i, $i \in \mathbb{Z}$, is a Λ-module in its own right where the module structure is m_2^N restricted to $m_2^N \colon \Lambda \oplus N^i \to N^i$. It is also worth noting that if $f \colon M \to N$ is a morphism and $m_1^M = 0 = m_1^N$, then for each $i \in \mathbb{Z}$, the map $f_1 \colon M^i \to N^i$ is a Λ-module morphism. A polydule with $m_1 = 0$ can informally be thought of as "homology + structure".

Our aim is to study the bounded derived category of finitely generated Λ-modules, $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, when Λ is a finite dimensional k-algebra. The category $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ is in this case equivalent to the full subcategory of $\mathcal{D}(\mathrm{Mod}\,\Lambda)$ consisting of objects with finite dimensional total homology. We get the following characterisation of the corresponding polydules.

Proposition 2.5. *Let Λ be a finite dimensional k-algebra. A Λ-polydule M is in the essential image of the composed functor $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \hookrightarrow \mathcal{D}(\mathrm{Mod}\,\Lambda) \xrightarrow{\sim} \mathcal{D}_\infty(\Lambda)$ if and only if it is quasi-isomorphic to a Λ-polydule M' with $m_1^{M'} = 0$ and $\dim_k M' < \infty$.*

We denote by $\mathcal{D}_\infty^f(\Lambda)$ the full subcategory of $\mathcal{D}_\infty(\Lambda)$ consisting of objects in this image. Obviously we have an equivalence of triangulated categories $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathcal{D}_\infty^f(\Lambda)$.

3. Description of variety

In this section we describe the geometric object we want to study. Suppose that k is an algebraically closed field and Λ is a finite dimensional k-algebra. We keep these assumptions for the rest of the paper. We fix dimensions in homology, non-zero only on a finite interval, and make a variety out of the possible Λ-polydule structures with $m_1 = 0$ and the given homology dimensions. Via the equivalence $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathcal{D}_\infty^f(\Lambda)$, we can view this as a parameter space of objects in the derived category with the given homology dimensions. There is a group of quasi-isomorphisms acting on this space. As noted by Keller [K, 4.3], what we have in this situation is an algebraic group acting on an algebraic variety. While this section is devoted to definitions, in section 5 we pursue our main goal, namely to find an algebraic characterisation of orbit closure (degeneration).

Suppose $\mathcal{B} = \{\nu_1 = 1, \nu_2, \ldots, \nu_t\}$ is a k-basis of Λ. We fix integers $a < b$ and a finite dimensional graded k-vector space $V = \oplus_{i=a}^b V^i$. We want to construct a variety $\mathcal{M}$ of possible A_∞-module structures on V. Let $d_i = \dim_k V^i$ for $a \leq i \leq b$, and let $\underline{d}$ denote the vector $\underline{d} = (d_a, \ldots, d_b)$. We fix bases for the graded parts of V, so that maps between them are represented by matrices.

A point in the variety $\mathcal{M} = \mathcal{M}_{\Lambda, \underline{d}}$ is a collection of matrices $M(S, i)$, one for each sequence $S = (\lambda_{|S|}, \ldots, \lambda_2, \lambda_1)$ of $1 \leq |S| \leq b - a + 1$ elements from $\mathcal{B}$ and i a number satisfying $a + |S| - 1 \leq i \leq b$. The matrix $M(S, i)$ is of size $d_{i-|S|+1} \times d_i$ and represents the map

$$m_{|S|+1}(\lambda_{|S|}, \ldots, \lambda_1, -) \colon V^i \to V^{i-|S|+1}.$$

We consider only strictly unital polydules, so if $1 \in S$ then

$$M(S, i) = \begin{cases} I & \text{if } |S| = 1 \\ 0 & \text{if } |S| > 1 \end{cases}$$

Given a sequence $S = (\lambda_{|S|}, \ldots, \lambda_1)$ and $1 \le l \le |S| - 1$, there are uniquely determined elements $c_r^{S,l} \in k$, $1 \le r \le t$ satisfying the expression

$$\lambda_{l+1} \lambda_l = \sum_{r=1}^{t} c_r^{S,l} \nu_r.$$

The map

$$m_{|S|}(\lambda_{|S|}, \ldots, \lambda_{l+2}, \lambda_{l+1} \lambda_l, \lambda_{l-1}, \ldots, \lambda_1, -) \colon V^i \to V^{i-|S|+2}$$

is represented by the matrix $\sum_{r=1}^{t} c_r^{S,l} M(S_{l/r}, i)$, where $S_{l/r}$ is the sequence

$$S_{l/r} = (\lambda_{|S|}, \ldots, \lambda_{l+2}, \nu_r, \lambda_{l-1}, \ldots, \lambda_1)$$

of length $|S| - 1$ obtained from S by deleting λ_{l+1} and λ_l and inserting ν_r in position l from the right.

The matrices $M(S, i)$ have to satisfy the relations

$$\sum_{S=[S',S'']} (-1)^{|S|(|S''|+1)} M(S', i - |S''| + 1) \cdot M(S'', i) = \sum_{l=1}^{|S|-1} \sum_{r=1}^{t} (-1)^{l-1} c_r^{S,l} \cdot M(S_{l/r}, i)$$

(This and other formulas in this section are obtained from the corresponding formulas in section 1.), where the sum on the left hand side is taken over all decompositions of S into two parts $S' = (\lambda_{|S|}, \ldots, \lambda_{|S''|+1})$ and $S'' = (\lambda_{|S''|}, \ldots, \lambda_1)$ with $|S'|, |S''| \ge 1$. We have one such relation for each pair (S, i) with $2 \le |S| \le b - a + 2$ and $a + |S| - 2 \le i \le b$.

Remark 3.1. Our construction of the variety $\mathcal{M}_{\Lambda,\underline{d}}$ depends on the fact that Λ has a finite k-basis. It might also be possible to construct a similar variety in some cases when Λ is not finite dimensional.

Next we describe the group $\mathcal{G} = \mathcal{G}_{\underline{d}}$ of quasi-isomorphisms. An element g in $\mathcal{G}$ is a collection of matrices $F(S, i)$, one for each sequence S of $0 \le |S| \le b - a$ elements and i a number satisfying $a + |S| \le i \le b$. Note that here we allow S to be the empty sequence $\varnothing$. The matrix $F(S, i)$ is of size $d_{i-|S|} \times d_i$ and is thought of as representing a map

$$f_{|S|+1}(\lambda_{|S|}, \ldots, \lambda_1, -) \colon V^i \to V^{i-|S|}.$$

We only want to consider strictly unital morphisms, so we add the condition that $F(S, i) = 0$ whenever $1 \in S$. Since $f_1 \colon V \to V$ is supposed to be a quasi-isomorphism, the matrices $F(\varnothing, i)$ must be invertible for all $a \le i \le b$. These are the only conditions for g to be an element in $\mathcal{G}$, there are no other conditions.

The product $g = (F(S, i))$ of two group elements $g' = (F'(S, i))$ and $g'' = (F''(S, i))$ is given by the rule

$$F(S, i) = \sum_{S=[S',S'']} (-1)^{|S'||S''|} F'(S', i - |S|) \cdot F''(S'', i).$$

The identity element e of this group is the element given by $F(\varnothing, i) = I_{d_i}$ for all i and $F(S, i) = 0$ for all $S \neq \varnothing$. The inverse $g^{-1} = (F'(S, i))$ of an element $g = (F(S, i))$ is given recursively by

$$F'(\varnothing, i) = F(\varnothing, i)^{-1}$$

and

$$F'(S, i) = \Big[\sum_{S=[S', S''], S'' \neq \varnothing} (-1)^{|S'||S''|+1} F'(S', i - |S|) \cdot F(S'', i) \Big] F(\varnothing, i)^{-1}$$

for all $S \neq \varnothing$.

We now describe the action of $\mathcal{G}$ on $\mathcal{M}$. The map $* \colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is given in such a way that if $g = (F(S, i))$ and $x = (M(S, i))$, then $x' = g * x$ is given by

$$\begin{aligned}
M'(S, i) = \Big[&\sum_{S=[S', S''], S'' \neq \varnothing} (-1)^{|S'|(|S''|+1)} F(S', i - |S''| + 1) \cdot M(S'', i) \\
&+ \sum_{l=1}^{|S|-1} \sum_{r=1}^{t} (-1)^{l-1} c_r^{S,l} \cdot F(S_{l/r}, i) \\
&+ \sum_{S=[S', S''], S', S'' \neq \varnothing} (-1)^{|S'||S''|+1} M'(S', i - |S|) \cdot F(S'', i) \Big] F(\varnothing, i)^{-1}.
\end{aligned}$$

The $\mathcal{G}$-orbits in $\mathcal{M}$ correspond to isomorphism classes of objects in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ with the given dimensions in homology. We define degeneration in $\mathcal{M}$ in the usual way as follows. Let M and N be two Λ-polydules with $m_1^M = 0 = m_1^N$ and the given homology dimensions. We use the same letters to denote the corresponding points in $\mathcal{M}$. If N belongs to the Zariski closure of the $\mathcal{G}$-orbit of M, in mathematical notation $N \in \overline{\mathcal{G} \cdot M}$, we say that M *degenerates to* N and we denote this fact by $M \leq_{\mathrm{deg}} N$. The relation $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of Λ-polydules with $m_1 = 0$ and the given homology dimensions.

4. Variations and simplifications

The variety $\mathcal{M}$ with its action by $\mathcal{G}$ has a rather complicated definition which is not so easy to work with in general. In this section we consider some special cases of algebras and polydules where we can define smaller and simpler spaces and group actions. We will illustrate these simplifications on some concrete examples in section 6.

Let $\mathrm{mod}(A, \underline{d}) \subseteq \mathcal{M}_{\Lambda, \underline{d}}$ be the subvariety consisting of polydules with $M(S, i) = 0$ for all S with $|S| > 1$. That is, each polydule in $\mathrm{mod}(\Lambda, \underline{d})$ is an graded Λ-module, and $\mathrm{mod}(\Lambda, \underline{d})$ is isomorphic to a product of module varieties

$$\mathrm{mod}(\Lambda, \underline{d}) \cong \prod_i \mathrm{mod}(\Lambda, d_i).$$

There is a projection map

$$\pi \colon \mathcal{M}_{\Lambda, \underline{d}} \longrightarrow \mathrm{mod}(\Lambda, \underline{d})$$

defined by $\pi(M(S,i)) = 0$ for all S with $|S| > 1$. Here M and N belong to the same fibre if and only if they have equal homology as graded modules. That is, $\pi(M) = H^*M$ computes the homology of a polydule.

The group $\mathsf{GL}_{\underline{d}} = \prod_i \mathsf{GL}_{d_i}$ acts on $\mathrm{mod}(\Lambda, \underline{d})$ by conjugation, and the orbits are in bijection with isomorphism classes of graded Λ-modules with graded dimension $\underline{d}$. There is a morphism

$$\mathcal{G} \longrightarrow \mathsf{GL}_{\underline{d}}$$

of algebraic groups mapping all $F(S,i)$ with $|S| > 0$ to zero. We let $\mathcal{G}$ act on $\mathrm{mod}(\Lambda, d)$ via this morphism. Then by the definition of the action of $\mathcal{G}$ on $\mathcal{M}$, the projection π is $\mathcal{G}$-equivariant.

4.1. Quivers with relations.

A very important combinatorial tool for studying modules over finite dimensional algebras is to use quivers with relations. We shall see that quivers are also useful when studying polydules.

Now assume that $\Lambda = kQ/I$ for a quiver $Q = (Q_0, Q_1)$ and an admissible ideal $I \subseteq kQ$. Then it is well known that the category of Λ-modules is equivalent to the category of representations of Q satisfying the relations in I (see [ARS] for an introduction to quivers and their representations). The change of point of view from modules to representations can be regarded as a change of underlying monoidal base category from k-modules to $(k^{\times r})$-modules, where $r = |Q_0|$. With the new base category the general theory in [L-H] is still valid, and a polydule in the new sense is the same a graded representation of Q with an A_∞-structure respecting the quiver. We call such a graded representation a *polyrepresentation*. From the general theory [L-H] it follows that the derived category of polyrepresentations is triangle equivalent to the derived category of representations, which we know is equivalent to the derived category of Λ-modules.

We will now define a variety of polyrepresentations. We fix not only dimensions in homology, but dimensions vectors. So the given data is a sequence $\underline{e} = (\underline{e}^a, \dots, \underline{e}^b)$ of dimension vectors $\underline{e}^i = (e^i_1, \dots, e^i_r)$, $a \le i \le b$. Let $\underline{d} = (d_a, \dots, d_b)$, where $d_i = \sum_{j \in Q_0} e^i_j$. Assume a basis for Λ is chosen that consists only of paths. We define a subvariety

$$\mathrm{Rep}_\infty(\Lambda, \underline{e}) \subseteq \mathcal{M}_{\Lambda, \underline{d}}$$

as follows. For each point $x = (M(S,i)) \in \mathcal{M}_{\Lambda, \underline{d}}$, divide each matrix $M(S,i)$ into blocks $C_{j,l}(S,i)$, $1 \le j, l \le r$, of size $e^{i-|S|+1}_l \times e^i_j$. The subvariety $\mathrm{Rep}_\infty(\Lambda, \underline{e})$ consists of all $x \in \mathcal{M}_{\Lambda, \underline{d}}$ which satisfy all of the following three conditions.

   (i) $M(S,i) = 0$ if the composition $\lambda_{|S|} \cdots \lambda_1$ of the paths in S is not a path in Q.
   (ii) $C_{j,l}(S,i) = 0$ if the composition $\lambda_{|S|} \cdots \lambda_1$ of the paths in S is a path in Q starting in vertex s and ending in vertex t and $(j,l) \ne (s,t)$.
   (iii) $M(S,i) = 0$ if $|S| > 1$ and S contains a path of length zero.

The variety $\mathrm{Rep}_\infty(\Lambda, \underline{e})$ is the variety of polyrepresentations with sequence of dimension vectors $\underline{e} = (\underline{e}^a, \dots, \underline{e}^b)$.

The image of $\mathrm{Rep}_\infty(\Lambda, \underline{e})$ under π,

$$\pi(\mathrm{Rep}_\infty(\Lambda, \underline{e})) = \mathrm{Rep}(\Lambda, \underline{e}) \subseteq \mathrm{mod}(\Lambda, \underline{d}),$$

is the variety of graded representations of Q satisfying the relations in I with sequence of dimension vectors $\underline{e} = (\underline{e}^a, \ldots, \underline{e}^b)$. For each $a \leq i \leq b$, a further projection to the ith component gives

$$\mathrm{Rep}(\Lambda, \underline{e}^i) \subseteq \mathrm{mod}(\Lambda, d_i),$$

the variety of representations of Q satisfying the relations in I with dimension vector $\underline{e}^i$.

For each element $g = (F(S, i)) \in \mathcal{G}_{\underline{d}}$, divide each matrix $F(S, i)$ into blocks $D_{j,l}(S, i)$, $1 \leq j, l \leq r$, of size $e_l^{i-|S|} \times e_j^i$. Let $\mathcal{G}_Q = \mathcal{G}_{Q,\underline{e}}$ be the subgroup of $\mathcal{G}_{\underline{d}}$ consisting of the elements $g \in \mathcal{G}_{\underline{d}}$ which satisfy all of the following four conditions.

(i) $D_{j,l}(\varnothing, i) = 0$ if $j \neq l$.

(ii) $F(S, i) = 0$ if $|S| > 0$ and the composition $\lambda_{|S|} \cdots \lambda_1$ of the paths in S is not a path in Q.

(iii) $D_{j,l}(S, i) = 0$ if $|S| > 0$, the composition $\lambda_{|S|} \cdots \lambda_1$ of the paths in S is a path in Q starting in vertex s and ending in vertex t and $(j, l) \neq (s, t)$.

(iv) $F(S, i) = 0$ if $|S| > 0$ and S contains a path of length zero.

The group $\mathcal{G}_Q$ is the group of quasi-isomorphisms which respect the quiver. By the general theory [L-H], we know that the $\mathcal{G}_Q$-orbits in $\mathrm{Rep}_\infty(\Lambda, \underline{e})$ correspond to quasi-isomorphism classes of polyrepresentations and therefore to isomorphism classes of objects in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$. Moreover, $\mathcal{G} \cdot \mathrm{Rep}_\infty(\Lambda, \underline{e})$ is a connected component of $\mathcal{M}_{\Lambda,\underline{d}}$, and each connected component is obtained in this way. This follows, since each polydule degenerates to its homology, and each module degenerates to the direct sum of its composition factors. We give a short argument showing that a polydule degenerates to its homology.

Lemma 4.1. *Let $M \in \mathcal{M}_{\Lambda,\underline{d}}$. Then $M \leq_{\mathrm{deg}} H^*(M)$.*

Proof. Let M_t be the polydule given by $M_t(S, i) = t^{|S|-1} M(S, i)$ for all $|S| > 1$. We have a strict isomorphism $M_t \cong M$ for all $t \neq 0$ and $M_0 = H^*(M)$. Therefore $M \leq_{\mathrm{deg}} H^*(M)$. $\qquad\square$

Assuming for a moment the results of the next section, we now show that degeneration of polyrepresentations is equivalent to degeneration of polydules.

Lemma 4.2. *Let $M, N \in \mathrm{Rep}_\infty(\Lambda, \underline{e})$. Then $M \leq_{\mathrm{deg}} N$ if and only if $N \in \overline{\mathcal{G}_Q \cdot M}$.*

Proof. Assume $N \in \overline{\mathcal{G}_Q \cdot M}$. We have $\mathcal{G}_Q \cdot M \subseteq \mathcal{G} \cdot M$ and therefore $\overline{\mathcal{G}_Q \cdot M} \subseteq \overline{\mathcal{G} \cdot M}$. Hence $M \leq_{\mathrm{deg}} N$.

Now assume that $M \leq_{\mathrm{deg}} N$. So there exists a polydule Z in $\mathcal{D}_\infty^f(\Lambda)$ and an exact triangle

$$Z \longrightarrow M \oplus Z \longrightarrow N \longrightarrow Z[1]$$

by Theorem 5.2. We may assume that Z is a polyrepresentation. Now, by the proof of Theorem 5.4 there exists a family of representations M_t in $\overline{\mathcal{G}_Q \cdot M}$, such that $M_t \cong M$ for all except a finite number of t, and $M_0 \cong N$. This shows that $N \in \overline{\mathcal{G}_Q \cdot M}$. $\qquad\square$

4.2. **Polydules with fixed homology.** Let $X \in \mathrm{mod}(\Lambda, \underline{d})$ be a graded module with graded dimension $\underline{d}$. Let $\mathcal{M}_X = \pi^{-1}(X)$ be the variety of polydules with fixed homology X. Let $\mathcal{G}_X$ be the setwise stabilizer of $\mathcal{M}_X$, consisting of group elements with $F(\varnothing, i) \in \mathrm{Aut}_\Lambda(H^i(X))$. There is a bijection between the orbits under the induced action of $\mathcal{G}_X$ on $\mathcal{M}_X$ and the isomorphism classes of polydules with homology equal to X. The variety $\mathcal{M}_X$ is connected with unique closed orbit the orbit of X. The following lemma may be proven similar to Lemma 4.2 above. We skip the details.

Lemma 4.3. *Let $M, N \in \mathcal{M}_X$. Then $M \leq_{\mathrm{deg}} N$ if and only if $N \in \overline{\mathcal{G}_X \cdot M}$.*

As a special case we can consider polydules with fixed homology in two degrees. That is, $X = M[a] \oplus N[b]$ for two Λ-modules M and N and degrees a and b. Then the equations defining $\mathcal{M}_X$ are affine equations, and therefore $\mathcal{M}_X$ is isomorphic to an affine space. The following lemma is a useful application of varieties of polydules to the problem of checking the vanishing of extension spaces between Λ-modules.

Lemma 4.4. *Let M and N be two finitely generated Λ-modules and let $X = M \oplus N[i]$, where $i \geq 0$. Then $\mathrm{Ext}_\Lambda^{i+1}(M, N) = 0$ if and only if $\mathcal{G}_X \cdot X = \mathcal{M}_X$.*

Proof. We have $\mathrm{Ext}_\Lambda^{i+1}(M, N) = \mathrm{Hom}(M, N[i+1]) \simeq \mathrm{Hom}(M[-1], N[i])$. If $f \colon M[-1] \to N[i]$ is a morphism, then the cone of f is in $\mathcal{M}_X$, and every point in $\mathcal{M}_X$ can be obtained as the cone of such a morphism. The cone of f is quasi-isomorphic to X if and only if f is null-homotopic. The lemma follows. $\square$

Suppose $\Lambda = kQ/I$ for a quiver $Q = (Q_0, Q_1)$ and an admissible ideal $I \subseteq kQ$. We can combine the constructions in this section and consider polyrepresentations with fixed homology. Let $X \in \mathrm{Rep}(\Lambda, \underline{e})$ be a graded representation with sequence of dimension vectors $\underline{e} = (\underline{e}^a, \ldots, \underline{e}^b)$. Then define

$$\mathcal{M}_X^Q = \mathcal{M}_X \cap \mathrm{Rep}_\infty(\Lambda, \underline{e}),$$

or equivalently $\mathcal{M}_X^Q = (\pi')^{-1}(X)$, where π' is the restricted function $\pi' = \pi|_{\mathrm{Rep}_\infty(\Lambda, \underline{e})} \colon \mathrm{Rep}_\infty(\Lambda, \underline{e}) \to \mathrm{Rep}(\Lambda, \underline{e})$. There is a group $\mathcal{G}_X^Q = \mathcal{G}_X \cap \mathcal{G}_{Q, \underline{e}}$ acting on $\mathcal{M}_X^Q$ and orbits correspond to isomorphism classes of polyrepresentations. Analogues of Lemmas 4.3 and 4.4 hold in this situation.

5. Characterisation of orbit closure

In this section we prove our main result, an algebraic characterisation of degeneration (orbit closure) in the variety $\mathcal{M}_{\Lambda, \underline{d}}$. Keeping in mind the equivalence $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathcal{D}_\infty^f(\Lambda)$, this should be seen as a derived category version of a celebrated theorem by Zwara [Z] and Riedtmann [R]. In fact the Riedtmann-Zwara Theorem is a special case of the result we prove here.

In [GO'H] the authors gave a very general theorem characterising orbit closures in varieties. It takes the following form when applied to our situation.

Theorem 5.1. *Let $M, N \in \mathcal{M}_{\Lambda, \underline{d}}$. Then $M \leq_{\mathrm{deg}} N$ if and only if there is a discrete valuation k-algebra R (residue field k, quotient field K finitely*

generated and of transcendence degree one over k) and a $\Lambda \otimes_k R$-polydule structure with $m_1 = 0$ on the graded R-module $Q = \oplus_{i=a}^b R^{d_i}$ such that $Q \otimes_R k = N$ (via the identification $R \otimes_R k = k$) and $Q \otimes_R K \simeq M \otimes_k K$ as $\Lambda \otimes_k K$-polydules.

Proof. The statement follows from [GO'H, 1.2].   □

Using this as a starting point, we can prove the "Zwara direction" of our characterisation. Zwara [Z] was the first to prove the analogous statement for module varieties. Yoshino [Y] defined module degeneration for any k-algebra, also when there is no underlying variety, and showed that Zwara's result was still valid in this generality [Y, 2.2]. Our proof, although completely different due to the triangulated setting, borrows some ideas from [Y].

Theorem 5.2 ("Zwara direction"). *Let $M, N \in \mathcal{M}_{\Lambda,\underline{d}}$. If $M \leq_{\deg} N$, then there is an object Z in $\mathcal{D}_\infty^f(\Lambda)$ and an exact triangle*

$$Z \to M \oplus Z \to N \to Z[1].$$

Proof. Suppose $M \leq_{\deg} N$. Let R be a discrete valuation k-algebra as in Theorem 5.1 and let $Q = \oplus_{i=a}^b R^{d_i}$ be a $\Lambda \otimes_k R$-polydule such that $m_1^Q = 0$, $Q \otimes_R k = N$ and $Q \otimes_R K \simeq M \otimes_k K$ as $\Lambda \otimes_k K$-polydules. Let t denote a generator of the maximal ideal in R.

Let $i \colon M \otimes_k R \to M \otimes_k K$ be the strict inclusion of $\Lambda \otimes_k R$-polydules and let f be the composition

$$M \otimes_k R \xrightarrow{i} M \otimes_k K \xrightarrow{\sim} Q \otimes_R K.$$

Since $M \otimes_k R$ and $Q \otimes_R K$ have support in only a finite number of degrees, we know that $f_i = 0$ for $i >> 0$. Since in addition $M \otimes_k R$ is finitely generated as an R-module, there is an n such that the morphism f factors through the strict inclusion $Q \otimes_R (1/t^n)R \to Q \otimes_R K$. There is a strict isomorphism $Q \otimes_R (1/t^n)R \simeq Q$. We have the following morphism of triangles in $\mathcal{D}_\infty(\Lambda)$.

$$
\begin{array}{ccccccc}
M \otimes_k R & \xrightarrow{\ g\ } & Q & \longrightarrow & \operatorname{cone} g & \xrightarrow{\ v\ } & (M \otimes_k R)[1] \\
\Big\| & & \Big\downarrow & & \Big\downarrow & & \Big\| \\
M \otimes_k R & \xrightarrow{\ f\ } & Q \otimes_R K & \longrightarrow & \operatorname{cone} f & \xrightarrow{\ u\ } & (M \otimes_k R)[1]
\end{array}
$$

Since f is a section, we have $u = 0$ by [H, I.4.1]. Since $u = 0$, also $v = 0$ and g is a section. It follows that $g_1 \colon M \otimes_k R \to Q$ is injective. Since $M \otimes_k R$ and Q have the same rank over R, it follows by the construction of the cone that $\operatorname{cone} g$ has finite dimensional total homology.

Now consider the strict inclusion of Λ-polydules $j \colon M \otimes_k tR \to M \otimes_k R$. From the exact sequence of vector spaces $0 \to tR \to R \to k \to 0$, we get an exact sequence of Λ-polydules $0 \to M \otimes_k tR \xrightarrow{j} M \otimes_k R \to M \to 0$, so $M \simeq \operatorname{cone} j$ in $\mathcal{D}_\infty(\Lambda)$. By the octahedral axiom, we get the following diagram of exact (marked by $\triangle$) and commuting triangles in $\mathcal{D}_\infty(\Lambda)$.
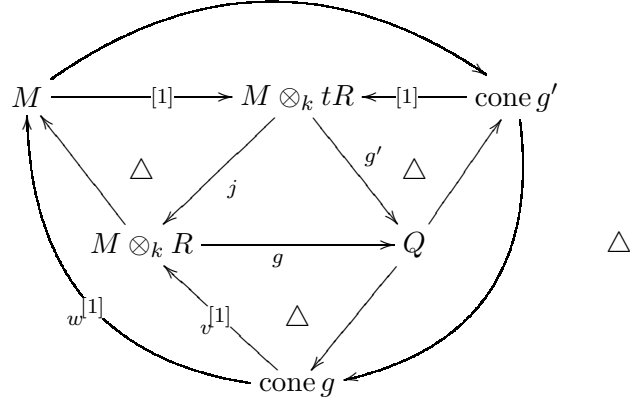
So there is an exact triangle $M \to \mathrm{cone}\, g' \to \mathrm{cone}\, g \xrightarrow{w} M[1]$. Since $v = 0$, also $w = 0$ and $\mathrm{cone}\, g' \simeq M \oplus \mathrm{cone}\, g$.

From the commutative square

$$
\begin{array}{ccc}
M \otimes_k R & \xrightarrow{\ g\ } & Q \\
\downarrow{\scriptstyle \tilde{t}} & & \downarrow{\scriptstyle \tilde{t}} \\
M \otimes_k tR & \xrightarrow{\ g'\ } & Q
\end{array} \ ,
$$

where $\tilde{t}$ denotes multiplication by t, we get the commutative diagram

where the rows and columns are exact triangles [W, Ex.10.2.6.]. We see that $\mathrm{cone}\, h \simeq Q \otimes_R k = N$, and there is an exact triangle

$$\mathrm{cone}\, g \xrightarrow{h} \mathrm{cone}\, g' \to \mathrm{cone}\, h \to (\mathrm{cone}\, g)[1]$$

in $\mathcal{D}_\infty(\Lambda)$. We put $Z = \mathrm{cone}\, g$, which is an object in $\mathcal{D}_\infty^f(\Lambda)$. The triangle above is isomorphic to $Z \to M \oplus Z \to N \to Z[1]$. $\qquad\square$

In order to prove the converse of this theorem, we first make the useful observation that the existence of an exact triangle of the form in question implies the existence of a certain exact sequence of polydules.

Lemma 5.3. *Let $M, N \in \mathcal{M}_{\Lambda,\underline{d}}$. Suppose there is an exact triangle of the form $Z \to M \oplus Z \to N \to Z[1]$ in $\mathcal{D}_\infty^f(\Lambda)$.*

Then there is an exact sequence of finite dimensional Λ-polydules of the form $0 \to Z \xrightarrow{f} M' \oplus Z' \xrightarrow{g} N \to 0$ with $m_1^Z = m_1^{Z'} = m_1^{M'} = 0$ and where $Z \simeq Z'$ and $M \simeq M'$. (These isomorphisms are possibly non-strict.)

Proof. Suppose there is a triangle $Z \to M \oplus Z \to N \xrightarrow{h} Z[1]$ in $\mathcal{D}_\infty^f(\Lambda)$. We may assume that Z is represented in such a way that $m_1^Z = 0$. The morphism $h[-1]\colon N[-1] \to Z$ gives rise to an exact sequence $0 \to Z \xrightarrow{f} \text{cone}(h[-1]) \xrightarrow{g} N \to 0$. Since $\dim_k M^i = \dim_k N^i$ for all $i \in \mathbb{Z}$, also $\dim_k(M \oplus Z)^i = \dim_k(\text{cone}(h[-1]))^i$ for all $i \in \mathbb{Z}$. The quasi-isomorphism $M \oplus Z \to \text{cone}(h[-1])$ must therefore be a (possibly non-strict) isomorphism and $m_1^{\text{cone}(h[-1])} = 0$. $\qquad\square$

We are now ready to prove the converse of Theorem 5.2. Our proof is similar to Riedtmann's original proof for module varieties [R, 3.4].

Theorem 5.4 ("Riedtmann direction"). *Let $M, N \in \mathcal{M}_{\Lambda,\underline{d}}$. If there is an exact triangle $Z \to M \oplus Z \to N \to Z[1]$ in $\mathcal{D}_\infty^f(\Lambda)$, then $M \leq_{\deg} N$.*

Proof. Suppose there is a triangle $Z \to M \oplus Z \to N \xrightarrow{h} Z[1]$ in $\mathcal{D}_\infty^f(\Lambda)$. From the previous lemma it follows that there is an exact sequence of finite dimensional Λ-polydules of the form $0 \to Z \xrightarrow{f} M' \oplus Z' \xrightarrow{g} N \to 0$ with $m_1^Z = m_1^{Z'} = m_1^{M'} = 0$ and where $v\colon Z \simeq Z'$ and $M \simeq M'$. We now show that this implies $M \leq_{\deg} N$.

Let $W = \text{Im}\, f$. For each $i \in Z$, choose a k-complement C^i of W^i in $M^i \oplus Z^i$, so $W^i \oplus C^i \simeq (M')^i \oplus (Z')^i$ as k-vector spaces. Let $\mathcal{F}$ be the set of all strict morphisms $\tilde{f}\colon Z \to M' \oplus Z'$ such that $\tilde{f}$ is injective and $(\text{Im}\,\tilde{f}) \cap C = 0$. These are open conditions, so $\mathcal{F}$ is a variety.

For each $\tilde{f} \in \mathcal{F}$ we define an A_∞-module structure on C and a strict morphism $\tilde{g}\colon M' \oplus Z' \to C$ such that $0 \to Z \xrightarrow{\tilde{f}} M' \oplus Z' \xrightarrow{\tilde{g}} C \to 0$ becomes an exact sequence. Let $\tilde{f} = \left(\begin{smallmatrix} \tilde{\phi} \\ \tilde{\psi} \end{smallmatrix}\right) \colon Z \to W \oplus C$. We assume $\tilde{f} \in \mathcal{F}$, so $\tilde{\phi}$ is an invertible map of graded vector spaces. We define

$$\tilde{g} = \begin{pmatrix} \tilde{\psi}(\tilde{\phi})^{-1} & \mathbf{1}_C \end{pmatrix} \colon W \oplus C \to C$$

and

$$m_n^C(\lambda_{n-1}, \ldots, \lambda_1, a) = \tilde{g}(m_n^{M' \oplus Z'}(\lambda_{n-1}, \ldots, \lambda_1, a))$$

for all $n \geq 2$, $a \in C$ and $\lambda_1, \ldots, \lambda_{n-1} \in \Lambda$. Since $\dim_k C^i = \dim_k N^i$ for all $i \in \mathbb{Z}$, this determines a morphism of varieties $\theta\colon \mathcal{F} \to \mathcal{M}_{\Lambda,\underline{d}}$. This morphism is defined in such a way that $\theta(\tilde{f})$ corresponds to a polydule quasi-isomorphic to cone $\tilde{f}$.

There is an open set $U \in \mathbb{A}_1$ containing 0 such that the morphism $f_{(t)} = f + t \cdot \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right)\colon Z \to M' \oplus Z'$ belongs to $\mathcal{F}$ whenever $t \in U$. Except for a finite number of points in U, the second component of $f_{(t)}$ has full rank and cone $f_{(t)} \simeq M' \simeq M$. If $t = 0$, then cone $f_{(0)} = \text{cone}\, f \simeq N$. So $M \leq_{\deg} N$. $\qquad\square$

Combining the results so far in this section we get exactly the algebraic characterisation of degeneration we were aiming for.

Theorem 5.5. *Let $M, N \in \mathcal{M}_{\Lambda,\underline{d}}$. Then the following are equivalent.*

(a) *$M \leq_{\deg} N$.*

(b) *There is an object Z in $\mathcal{D}_\infty^f(\Lambda)$ and an exact triangle*

$$Z \to M \oplus Z \to N \to Z[1].$$

(c) *There is an exact sequence of finite dimensional Λ-polydules of the form*

$$0 \to Z \to M' \oplus Z' \to N \to 0$$

with $m_1^Z = m_1^{Z'} = m_1^{M'} = 0$ and where $Z \simeq Z'$ and $M \simeq M'$.

If $a = b = 0$ and $\underline{d} = (d)$, then $\mathcal{M}_{\Lambda,\underline{d}}$ is equal to the module variety $\mathrm{mod}(\Lambda, d)$. We get the Riedtmann-Zwara Theorem ([Z], [R]) as a corollary.

Corollary 5.6 (Riedtmann-Zwara Theorem). *Let Λ be a finite dimensional k-algebra and let M and N be two d-dimensional Λ-modules. Then $M \leq_{\deg} N$ if and only if there exists a finite dimensional Λ-module Y and a short-exact sequence $0 \to Y \to M \oplus Y \to N \to 0$.*

Proof. If $v \colon Z \to Z'$ is an isomorphism of polydules, then it induces a bijective Λ-morphism (so a Λ-isomorphism) $v_1 \colon Z^0 \to (Z')^0$. From the equivalence between (a) and (c) in the theorem it follows that $M \leq_{\deg} N$ if and only if there exists a finite dimensional Λ-module Z^0 and an exact sequence of Λ-modules $0 \to Z^0 \to M \oplus Z^0 \to N \to 0$. $\qquad\square$

In [JSZ], another theory of degeneration in derived categories was developed. There degenerations are studied in the space $\mathsf{comproj}^{\underline{d}}$, a space which parameterise right bounded complexes of projective modules with so-called dimension array $\underline{d}$. There is a group G acting on this space and G-orbits correspond to isomorphism classes in $\mathcal{D}(\mathrm{mod}\,\Lambda)$ where the dimension array appears (see [JSZ] for details). One then defines $M \leq_{\mathrm{top}} N$ if N is in the closure of the G-orbit of M. This concept of degeneration is compatible with the one in the present paper.

Corollary 5.7. *Let X and Y be two right bounded complexes of finitely generated projective Λ-modules with same dimension array $\underline{d}$. Suppose $\dim_k H^i X = \dim_k H^i Y$ for all $i \in \mathbb{Z}$ and that $\sum_{i \in \mathbb{Z}} \dim_k H^i X < \infty$. Let M_X and M_Y be Λ-polydules with $m_1^{M_X} = 0 = m_1^{M_Y}$ which are quasi-isomorphic to X and Y respectively.*
Then $M_X \leq_{\deg} M_Y$ if and only if $X \leq_{\mathrm{top}} Y$.

Proof. The relation $X \leq_{\mathrm{top}} Y$ can be characterised by the existence of a right bounded complex Z in $\mathcal{D}(\mathrm{mod}\,\Lambda)$ and an exact triangle $Z \to X \oplus Z \to Y \to Z[1]$ (see [JSZ]). Since both X and Y have bounded homology, also Z can be chosen in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$. Now the statement follows from Theorem 5.5 and the equivalence of triangulated categories $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathcal{D}_\infty^f(\Lambda)$. $\qquad\square$

6. EXAMPLES

We now give some examples illustrating the main theorem and also the different varieties constructed in section 4.

6.1. **Example 1.** Let $M \in \mathcal{M}_{\Lambda,\underline{d}}$. From Lemma 4.1 we know that $M \leq_{\deg} H^*M$. We find a polydule Z and a triangle $Z \longrightarrow M \oplus Z \longrightarrow H^*M \longrightarrow Z[1]$ in this case. Assume that H^iM is non-zero only in an interval $s \leq i \leq t$. We use the notation $Z_i = \tau^{\leq i}M$ for the truncated polydules. (Truncation is straightforward when $m_1^M = 0$.) We have an exact triangle

$$\eta_{t-1} \colon Z_{t-1} \longrightarrow M \longrightarrow H^tM \longrightarrow Z_{t-1}[1]$$

and by induction exact triangles

$$\eta_i \colon Z_i \longrightarrow Z_{i+1} \longrightarrow H^{i+1}M \longrightarrow Z_i[1]$$

for $s - 1 \leq i < t$. Then by taking the direct sum $\eta = \eta_{t-1} \oplus \cdots \oplus \eta_{s-1}$ of exact triangles and letting $Z = Z_{t-1} \oplus Z_{t-2} \oplus \cdots \oplus Z_s \oplus Z_{s-1}$, where $Z_{s-1} = 0$, we have the exact triangle

$$\eta \colon Z \longrightarrow M \oplus Z \longrightarrow H^*M \longrightarrow Z[1].$$

6.2. **Example 2.** Let A be the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$$

with relations $\beta\alpha = 0 = \delta\gamma$. The global dimension of A is 2. Let

$$X_1 = S_5[2] \oplus S_3[1] \oplus S_1,$$

where S_i denotes the simple module corresponding to vertex i. We compute the variety $\mathcal{M}_{X_1}^Q$ of polyrepresentations with homology module X_1, as defined in subsection 4.2. Let

$$x = M((\beta, \alpha), 0),$$
$$y = M((\delta, \gamma), -1),$$
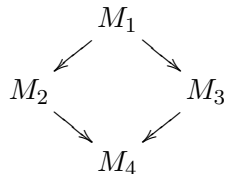$$z = M((\delta, \gamma\beta, \alpha), 0).$$

(We identify $M(S, i)$ with its unique non-zero block, a natural thing to do when working with polyrepresentations.) Then $\mathcal{M}_{X_1}^Q$ is the subvariety of $\mathbb{A}^3$ given by the equation $z = yx$. The group

$$\mathcal{G}_{X_1}^Q = (k^*)^3$$

acts by $(a, b, c)(x, y, z) = (bxc^{-1}, ayb^{-1}, azc^{-1})$. There is only the action given by change of basis in this example. There are four orbits given by the polydules

$$M_1 \colon x = 1, y = 1, z = 1,$$
$$M_2 \colon x = 1, y = 0, z = 0,$$
$$M_3 \colon x = 0, y = 1, z = 0,$$
$$M_4 \colon x = 0, y = 0, z = 0.$$

The Hasse diagram of degeneration is given by

$$
\begin{array}{ccc}
 & M_1 & \\
\swarrow & & \searrow \\
M_2 & & M_3 \\
\searrow & & \swarrow \\
 & M_4 &
\end{array}
$$

Consider the degeneration $M_1 \leq_{\deg} M_3$. Let $\Lambda = \tau^{\leq -1} M_3$. We have a decomposition $M_3 = S_1 \oplus L$. There is an exact sequence of polydules $0 \to L \xrightarrow{f} M_1 \xrightarrow{g} S_1 \to 0$. Using a standard trick, we get an exact sequence of polydules

$$0 \to L \xrightarrow{(f\ 0)} M_1 \oplus L \xrightarrow{h} M_3 \to 0$$

where $h = \begin{pmatrix} g & 0 \\ 0 & 1_L \end{pmatrix} : M_1 \oplus L \longrightarrow S_1 \oplus L$. This is a sequence of the same form as in Theorem 5.5.

6.3. **Example 3.** Let A be as in the previous example. Let $\underline{e}^{-1} = (1,0,0,0,0)$ and $\underline{e}^0 = (0,0,1,1,0)$ and be two dimension vectors and let $\underline{e} = (\underline{e}^{-1}, \underline{e}^0)$. We consider the variety $\mathrm{Rep}_\infty(A, \underline{e})$ defined in subsection 4.1. Let

$$x = M((\beta, \alpha), 0),$$
$$y = M((\gamma\beta, \alpha), 0),$$
$$z = M((\gamma), -1).$$

Then $\mathrm{Rep}_\infty(A, \underline{e})$ is given by the triples (x, y, z) with $y = -zx$. The group $G_{Q,\underline{e}} = (k^*)^3$ acts by $(a,b,c)(x,y,z) = (bxa^{-1}, cya^{-1}, czb^{-1})$. There is only the action given by change of basis in this example. There are four orbits given by the polydules

$$M_1 : x = 1, y = -1, z = 1,$$
$$M_2 : x = 1, y = 0, z = 0,$$
$$M_3 : x = 0, y = 0, z = 1,$$
$$M_4 : x = 0, y = 0, z = 0.$$

The Hasse diagram of degeneration is given by
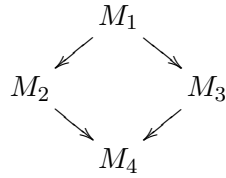
Let M be the representation

$$0 \longrightarrow 0 \longrightarrow k \xrightarrow{1} k \longrightarrow 0 \ .$$

Then

$$H^*(M_1) = S_1 \oplus M[1] = H^*(M_3)$$

and

$$H^*(M_2) = H^*(M_4) = S_1 \oplus (S_3 \oplus S_4)[1].$$

6.4. **Example 4.** Let A be the algebra of the previous two examples. Let $X_2 = S_j[2] \oplus S_1$ for some $j = 1,2,3,4,5$. Since the global dimension of A is 2, we see that $\mathrm{Ext}^3_A(S_1, S_j) = 0$. So according to Lemma 4.4 (or more precisely its version for polyrepresentations) we have $\mathcal{G}^Q_{X_2} \cdot X_2 = \mathcal{M}^Q_{X_2}$. We show this in detail. Since the multiplication m_4 takes three arguments from A, we must have $j = 4$ or 5 in order to have a non-trivial variety $\mathcal{M}^Q_{X_2}$. If $j = 5$, then $\mathcal{M}^Q_{X_2}$ is given by $x = M((\delta, \gamma\beta, \alpha), 0)$ with the equation $x = 0$,

which gives the trivial variety. If $j = 4$, then $\mathcal{M}_{X_2}^Q$ is given by the affine line $x = M((\gamma, \beta, \alpha), 0)$ with no relations. The group $\mathcal{G}_{X_2}^Q = (k^*)^2 \times k$ acts by

$$(a, b, c) \cdot t = (ax + c)b^{-1}.$$

Hence there is only one orbit. This shows that $\mathcal{G}_{X_2}^Q \cdot X_2 = \mathcal{M}_{X_2}^Q$.

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